The Linear Complementarity Problem, Sufficient Matrices, and the Criss-Cross Method

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ABSTRACT

Specially structured linear complementarity problems (LCPs) and their solution by the criss-cross method are examined. The criss-cross method is known to be finite for LCPs with positive semidefinite bisymmetric matrices and with P-matrices. It is also a simple finite algorithm for oriented matroid programming problems. Recently Cottle, Pang, and Venkateswaran identified the class of (column, row) sufficient matrices. They showed that sufficient matrices are a common generalization of P- and PSD matrices. Cottle also showed that the principal pivoting method (with a clever modification) can be applied to row sufficient LCPs. In this paper the finiteness of the criss-cross method for sufficient LCPs is proved. Further it is shown that a matrix is sufficient if and only if the criss-cross method processes all the LCPs defined by this matrix and all the LCPs defined by the transpose of this matrix and any parameter vector.

1. INTRODUCTION

In this paper we consider the linear complementarity problem (LCP). This problem asks for n-dimensional vectors w and z such that

$$-Mz + w = q, \qquad z \geqslant 0, \quad w \geqslant 0, \quad z^T w = 0, \tag{1}$$

where q is an n-dimensional vector, and M is an $n \times n$ matrix. We will refer

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to this problem by the pair (q, M). The solvability of (q, M) depends on certain properties of the coefficient matrix M. If matrix M is (column, row) sufficient, then we will simply speak about a (column, row) sufficient LCP. If we have a vector z (x), then Z (X) will denote the diagonal matrix with diagonal elements z_i (x_i) for all i. The unit matrix is denoted by E, and e_i denotes the ith unit vector with appropriate dimension. Finally, m_{ij} will denote the coefficient in row i and column j of the matrix M.

The LCP is one of the most widely studied problems of mathematical programming. Several methods have been developed for solving LCPs in the last decades (see e.g. Aganagić and Cottle [1], Cottle [3], Cottle and Dantzig [5], Lemke [16], Van der Heyden [23]). These methods utilize different pivot rules. There also exist several nonpivot methods. An excellent survey of the existing methods and the classification of matrices for LCPs can be found in Murty's [18] book. Nowadays the LCP is a subject of research on different (though interacting) approaches:

(1) *Polynomial methods.* First Kozlov et al. [15] gave a polynomial method for quadratic programming (QP) (a special LCP) by generalizing the ellipsoid method for this problem. Since then several papers have appeared presenting polynomial time interior point methods for quadratic programming (see e.g. [8, 25]) and the LCP (see e.g. [13, 14, 26, 27]).

(2) *Combinatorial abstraction.* Todd [22] and Morris and Todd [17] gave a combinatorial generalization of QP and LCP by formulating the QP problem and LCP of oriented matroids. Todd [22] generalized Lemke's [16] method as well. Klafszky and Terlaky [11, 12] generalized the criss-cross method [20, 21, 24], and Fukuda and Terlaky [9] gave finite pivoting rules for QP. The sufficiency property is also generalized to oriented matroids by Fukuda and Terlaky [10]. There the criss-cross method is also generalized to solve sufficient oriented matroid LCPs. To generalize the characterization theorems of this paper still remains a subject of further research.

(3) *Identification of matrix classes.* The class of (column, row) sufficient matrices was introduced by Cottle et al. [7]. They showed that (column, row) sufficient matrices are common generalizations of P-matrices (i.e. matrices with positive principal submatrices) and PSD matrices (positive semidefinite matrices). Later Cottle [4] generalized the principal pivoting method for row sufficient LCPs. Recently Cottle and Guu [6] gave another characterization for sufficient matrices.

This paper is somewhere on the border between the last two approaches. It examines sufficient LCPs and their solution by the criss-cross method. As we will see, the definition of (column, row) sufficient matrices relies essentially on sign properties, so this is a combinatorial characterization of matrix classes. The criss-cross method is a simple, purely combinatorial method, so the object of this paper is to characterize a matrix class by the finiteness of a combinatorial method. This object is fully reached by using the results of Cottle and Guu [6].

Up to now the criss-cross method was though to have been discovered first by Terlaky [20, 21] and later independently by Wang [24]. In the refereeing process one of the associate editors kindly called our attention to the unpublished work of Chang [2]. It turned out that a finite pivot rule as an extension of Murty's [19] scheme was presented on p. 49 of Chang's preprint. This extended Murty's scheme is equivalent to the QP criss-cross method presented by Klafszky and Terlaky [11], but the finiteness proof is completely different. As a consequence this paper can also be regarded as a further extension of Murty's scheme. In minimal index type methods there is no minimal ratio test. This cuts down the computational effort per iteration.

The criss-cross method is known to be finite for LCPs with positive semidefinite bisymmetric matrices [11, 2] and with P-matrices [19, 11]. It is also a simple finite algorithm for oriented matroid programming problems [12]. The properties that are necessary to guarantee the applicability and finiteness of the criss-cross method are studied in this paper. We will show that the criss-cross method is finite for sufficient LCPs. Further, it is also proved that a matrix M is sufficient if and only if the criss-cross method processes problems (q, M) and (q, M^T) with any parameter vector q. As for terminology, we say that the criss-cross method *processes* a problem if it finds a solution or detects infeasibility in a finite number of steps.

The paper is organized as follows. Section 2 contains a brief summary of the basic properties of (column, row) sufficient matrices. The criss-cross method is stated in Section 3, and the properties that are necessary to execute it and guarantee its finiteness are presented in Section 4. The characterization of the class of sufficient matrices by the criss-cross method is discussed in Section 5.

2. BASIC PROPERTIES OF SUFFICIENT MATRICES

The concept of (column, row) sufficient matrices was introduced by Cottle et al. [7]. For ease of understanding, the definition and basic properties of sufficient matrices are summarized here. The proofs and further details can be found in [7, 4, 6].

DEFINITION 1. A matrix M is called

(1) *row sufficient* if $XM^Tx \leqslant 0$ implies $XM^Tx = 0$ for every vector x (i.e., if $x_i(M^Tx)_i \leqslant 0$ for all i, then $x_i(M^Tx)_i = 0$ for every i);

(2) *column sufficient* if $XMx \leqslant 0$ implies $XMx = 0$ for every vector x;
(3) *sufficient* if it is both row and column sufficient.

This definition of (column, row) sufficient matrices closely relates to the well-known sign (non)reversibility property of matrices. Since this property is well established in oriented matroids, it is possible to generalize sufficiency to oriented matroids [10].

It has been proved that P- and PSD matrices are (row, column) sufficient matrices, but there are sufficient matrices that are neither P- nor PSD matrices. It is also known that the solution set of (q, M) is convex (polyhedral) if and only if matrix M is column sufficient. The following properties of (column, row) sufficient matrices (see [7, 4]) will be used in our discussions.

PROPOSITION 1. *Every principal rearrangement of a (column, row) sufficient matrix is (column, row) sufficient.*

PROPOSITION 2. *Let D be an invertible diagonal matrix. Then a matrix M is (column, row) sufficient if and only if DMD is (column, row) sufficient.*

PROPOSITION 3. *Every principal submatrix of a (column, row) sufficient matrix is (column, row) sufficient.*

PROPOSITION 4. *Both column and row sufficient matrices have nonnegative principal submatrices, and hence nonnegative diagonal elements.*

PROPOSITION 5.

(1) *Let M be row sufficient with $m_{ii} = 0$ for some i. If $m_{ij} \neq 0$ for some j, then $m_{ji} \neq 0$, and in this case $m_{ij}m_{ji} < 0$.*
(2) *Let M be column sufficient with $m_{ii} = 0$ for some i. If $m_{ji} \neq 0$ for some j, then $m_{ij} \neq 0$, and in this case $m_{ji}m_{ij} < 0$.*
(3) *Let M be sufficient with $m_{ii} = 0$ for some i. One has $m_{ij} \neq 0$ for some j if and only if $m_{ji} \neq 0$, and then $m_{ij}m_{ji} < 0$.*

Let a diagonal element m_{ii} be zero for for some i. Then as a consequence of Proposition 5 for (row, column) sufficient matrices we have:

(1) *For row sufficient matrices*: If $m_{ji} \geqslant 0$ for all j, then $m_{ij} \leqslant 0$ for all j. If $m_{ji} \leqslant 0$ for all j, then $m_{ij} \geqslant 0$ for all j.
(2) *For column sufficient matrices*: If $m_{ij} \geqslant 0$ for all j, then $m_{ji} \leqslant 0$ for all j. If $m_{ij} \leqslant 0$ for all j, then $m_{ji} \geqslant 0$ for all j.
(3) *For sufficient matrices*: $m_{ij} \leqslant 0$ for all j if and only if $m_{ji} \geqslant 0$ for all j. Moreover, $m_{ij} \geqslant 0$ for all j if and only if $m_{ji} \leqslant 0$ for all j.

PROPOSITION 6. *Any principal pivotal transform of a (column, row) sufficient matrix is (column, row) sufficient.*

The following results have been proved by Cottle and Guu [6].

PROPOSITION 7. *A 2×2 matrix M is sufficient if and only if for every principal pivotal transform $\overline{M}$ of M*

(1) $\overline{m}_{ii} \geqslant 0$ and
(2) for $i = 1, 2$, if $\overline{m}_{ii} = 0$, then either $\overline{m}_{ij} = \overline{m}_{ji} = 0$ or $\overline{m}_{ij}\overline{m}_{ji} < 0$ for $j \neq i$.

PROPOSITION 8. *A matrix M is sufficient if and only if every principal pivotal transform $\overline{M}$ of M is sufficient of order 2 (i.e., every 2×2 principal submatrix of $\overline{M}$ is sufficient).*

The criss-cross method will be defined in the next section. It will be shown that the criss-cross method is finite on sufficient LCPs. It is also proved that if the matrix M is not sufficient, then for some vector q the criss-cross method fails to solve (q, M) or (q, M^T). So the class of sufficient matrices can be characterized by the applicability and finiteness of the criss-cross method.

This section is closed by recalling the well-known *orthogonality property* of canonical tableaus: any row vector of a canonical tableau is orthogonal to any column vector of any dual canonical tableau (see e.g. [12, 20, 9]). Here a *row vector* means a vector which has the same dimension as the number of variables, and whose coordinates are identical with the corresponding coordinates of the actual row of the canonical tableau (the coordinates of the basic variables are 0 except for one, which is 1). A *column vector* of the dual canonical tableau means a vector with the same dimension as the row vector, whose coordinates are 0 at nonbasic positions except for one, which is -1, and whose coordinates in basic positions come from the tableau. The orthogonality property will play a crucial role in our discussions. Therefore we define it more precisely:

Let $\overline{T}$ be an arbitrary $m \times n$ matrix, and B a basis chosen from the column vectors of $\overline{T}$. Let J_B and $\overline{J}_B$ denote the sets of indices of the basic and nonbasic variables respectively (so $\{1, \ldots, n\} = J = J_B \cup \overline{J}_B$). Then the canonical tableau of $\overline{T}$ with respect to B contains coefficients τ_{ik}, where τ_{ik} is the coefficient of the basic vector $\bar{t}_i$ in the basic representation of vector $\bar{t}_k$, that is, $\bar{t}_k = \sum_{i \in J_B} \tau_{ik} \bar{t}_i$ for all $k \in J$. We proceed by introducing vectors $t_i \in R^n$, $i \in J$, as follows. If $i \in J_B$ then t_i is simply a row of the canonical tableau, namely the (unique) row which corresponds to basis vector $\bar{t}_i$. So $t_{ij} = \tau_{ij}$ for $j = 1, \ldots, n$. Note that $t_{ii} = 1$ and $t_{ij} = 0$ if $j \in J_B$ and $j \neq i$. If $i \in \overline{J}_B$, then we define t_i as a column of the *dual canonical tableau*. Then

t_i has the coordinates

$$t_{ij} = \begin{cases} \tau_{ji} & \text{if } j \in J_B, \\ -1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

The well-known orthogonality property is stated in the following lemma (see e.g. [9, 12, 20]).

LEMMA 1. *For any two bases B and B' we have that t_i is orthogonal to t'_k for all $i \in J_B$ and $k \in \bar{J}_{B'}$, where t_i is defined by the basis B and t'_k is defined by the basis B'.*

Proof. The orthogonality of the two vectors is obvious if $B = B'$. This implies that the subspace spanned by $\{t_i : i \in J_B\}$ is the orthogonal complement of the subspace spanned by $\{t_k : k \in \bar{J}_B\}$. Since pivoting (changing the basis) preserves the row space of canonical tableaus (the first subspace above), the orthogonal complement remains also the same. This implies the lemma. ∎

For better understanding let us consider the following simple numerical example: two basic tableaus that can be transformed into each other by a single pivot. The bases are $J_B = \{5, 4\}$ in the first tableau and $J_{B'} = \{2, 4\}$ in the second tableau:

	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}_3$	$\bar{t}_4$	$\bar{t}_5$
t_5	3	1	3	0	1
t_4	5	2	4	1	0

	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}_3$	$\bar{t}_4$	$\bar{t}_5$
t'_2	3	1	3	0	1
t'_4	-1	0	-2	1	-2

It is easy to check, for example, that from the first tableau we get $t_3 = (0, 0, -1, 4, 3)$ and from the second tableau $t'_4 = (-1, 0, -2, 1, -2)$. Obviously these vectors are orthogonal ($t_3^T t'_4 = 0$).

We will use this result, the so-called orthogonality property of canonical tableaus, for the matrix $\overline{T} = [-M, E, q]$. Here E provides a basis. In this tableau for $i \in J_B$ we have $t_i^T = (-m_{i\cdot}^T, e_i^T, q_i)$ and $t_k^T = (-e_k^T, -m_{\cdot k}^T, 0)$ for $k \in \bar{J}_B$, where $m_{i\cdot}$ denotes row i and $m_{\cdot k}$ denotes column k of the matrix M.

3. THE CRISS-CROSS METHOD FOR LCP'S

Let an LCP be given as it is presented in the Introduction. The initial basis is given by the matrix E, and the initial tableau is $[-M, E, q]$. A tableau is called *complementary* if the corresponding solution satisfies the complementarity condition. The above-defined initial tableau is complementary. For simplicity the nonbasic part of any complementary canonical tableau will also be denoted by $-M$ if no confusion is possible. Note that the nonbasic part of any complementary canonical tableau is a principal pivotal transform of the original matrix $-M$. We will say that our algorithm **STOP**s if the problem is processed, while **EXIT** is used if it fails to process the problem. The criss-cross method is defined as follows.

CRISS-CROSS METHOD.

Initialization:

Let the starting basis be defined by w, and let $w = q$, $z = 0$ be the initial solution.

The initial tableau is given by $[-M, E, q]$.

Pivot rule:

We have a complementary basis and the corresponding tableau.

Leaving variable selection:

Let $k := \min\{i : w_i < 0 \text{ or } z_i < 0\}$.

If there is no such k, then **STOP**; a feasible complementary solution has been found. (Without loss of generality we may assume that $w_k < 0$.)

Entering variable selection:

Diagonal pivot:

If $-m_{kk} < 0$, then make a diagonal pivot and repeat the procedure. (Here w_k leaves and z_k enters the basis.)

If $-m_{kk} > 0$, then **EXIT**.

If $-m_{kk} = 0$, select an exchange pivot.

Exchange pivot:

We know that $m_{kk} = 0$ in this case. Let $r := \min\{j : -m_{kj} < 0 \text{ or } -m_{jk} > 0\}$.

If there is no r, then **STOP**; LCP is infeasible.

If there is an r and $m_{rk}m_{kr} \geqslant 0$, then **EXIT**.

If there is an r and $m_{rk}m_{kr} < 0$, then make an exchange pivot on (r, k) and repeat the procedure. (Here w_k and z_r leave and z_k and w_r enter the basis.)

First of all note that for some problems the algorithm may **EXIT**. Some sufficient and some necessary properties that guarantee that this will not happen are discussed later on.

The algorithm is initialized with a complementary solution, and since it performs only diagonal and exchange pivots, complementarity is obviously preserved. If there is no leaving variable, then the current solution solves (q, M), since it is nonnegative and complementary. This property is independent of the special properties of M.

If there is no entering variable, then we have a nonnegative row with a negative solution coordinate. In this case there is no solution for (q, M). Indeed, if a solution existed, then one would have a nonnegative column (the solution column) for the corresponding tableau, which contradicts the orthogonality property (see e.g. [12, 9]).

The above remarks show that if the criss-cross method **STOP**s, then the conclusion (solution, infeasibility) easily follows. This implication is independent of the properties of the matrix M. We are interested in those properties of M which are necessary and sufficient to implement the criss-cross method successfully, i.e. provide the desired pivot in both of the diagonal and exchange pivot case (the algorithm does not **EXIT**) and guarantee its finiteness (prevent cycling).

We note that the usual form of the criss-cross method searches the row of the leaving variable for an exchange pivot, while here the corresponding column is searched as well. That makes no difference in the case of symmetric P- and PSD matrices or in the case of sufficient matrices (see Propositions 1–8 above), but in the case of nonsymmetric matrices this additional search makes the method more symmetric again.

4. SUFFICIENT AND NECESSARY PROPERTIES FOR THE FINITENESS OF THE CRISS-CROSS METHOD

Let $\mathscr{M}$ be the class of matrices such that for each $M \in \mathscr{M}$ and for each vector $q \in R^n$ the problem (q, M) is processed successfully by the criss-cross method (i.e. does not **EXIT** and does not cycle). We will first derive some necessary properties for $\mathscr{M}$. The matrix class $\mathscr{M}$ has to be closed with respect to principal pivot transformation, and the principal submatrices of every matrix $M \in \mathscr{M}$ must belong to $\mathscr{M}$ as well. (Propositions 3 and 6 state that the classes of column and row sufficient matrices are complete and closed with respect to principal pivotal transformation. We will refer to such a matrix class as a *closed complete class*.)

The first property guarantees that if a diagonal pivot is possible, then the entering variable will be nonnegative. The solution process goes in the "good direction."

PROPERTY 1. *If* $M \in \mathcal{M}$, *the diagonal elements of any principal pivotal transform of* $-M$ *are nonpositive.*

The second property is required to ensure the possibility of an exchange pivot if a diagonal pivot is not possible. In this case the complementary pair of the driving variable will enter at a nonnegative (feasible) level.

PROPERTY 2. *If* $-m_{kk} = 0$ *for some* k, *then* $-m_{kj} < 0$ *if and only if* $-m_{jk} > 0$ *for any* j.

If the above two properties hold, then the criss-cross method can perform a pivot in any situation, i.e. it stops if and only if the problem (q, M) has been processed. The only problem remains to prevent cycling.

A third property is required to guarantee the finiteness (to exclude the possibility of cycling) of the criss-cross method. We will say that two tableau types are *exclusive* for a (q, M) if at most one of them may exist for the given problem. The next property requires that four pair of tableau types are exclusive; the tableau types (called A, B, C and D) are defined by sign properties.

PROPERTY 3. *For a problem* (q, M) *the pairs of cases* **AB**, **CD**, **AC**, *and* **BD** *are exclusive for any index* $1 \leqslant k \leqslant n$:

A: *We have a complementary tableau with* $w_i \geqslant 0$, $z_i \geqslant 0$ *for* $i < k$, *and* $w_k = 0$, $z_k < 0$.

B: *We have a complementary tableau with* $w_i \geqslant 0$, $z_i \geqslant 0$ *for* $i < k$, *and* $w_k < 0$, $z_k = 0$.

C: *We have a complementary tableau with* $z_s < 0$ *for some* $s < k$, $m_{si} \geqslant 0$ *for* $i < k$, $m_{ss} = 0$, *and* $m_{sk} < 0$; *and symmetrically* $m_{is} \leqslant 0$ *for* $i < k$ *and* $m_{ks} > 0$.

D: *We have a complementary tableau with* $w_s < 0$ *for some* $s < k$, $m_{si} \geqslant 0$ *for* $i < k$, $m_{ss} = 0$, *and* $m_{sk} < 0$; *and symmetrically* $m_{is} \leqslant 0$ *for* $i < k$ *and* $m_{ks} > 0$.

The sign structures of the complementary tableaus associated with the four cases of Property 3 are demonstrated in Figure 1. The matrices are divided into parts according to the basic and nonbasic set of variables of z and w. Here we have assumed that $k = n$.

Further note that the only restrictive requirement in Property 3 is that tableau types **A** and **B** are exclusive; for the other three pairs of tableau types the exclusivity simply follows from the orthogonality property.

Let $\mathcal{M}'$ denote the class of matrices for which Properties 1, 2, and 3 hold for any vector q and which is complete with respect to these properties.

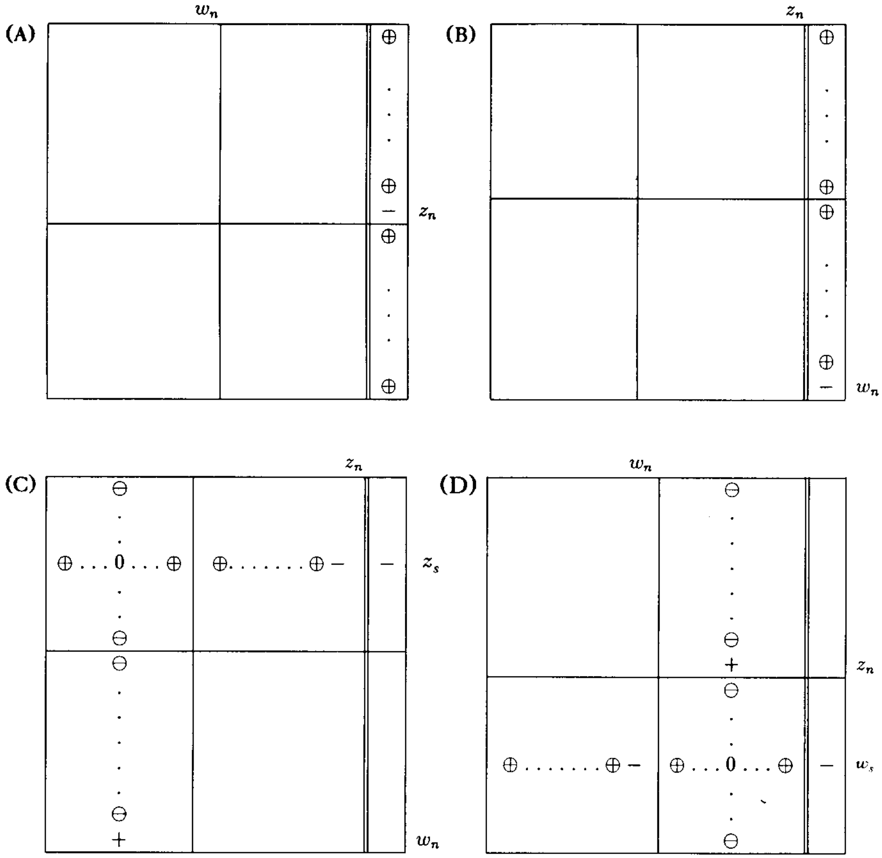

FIG. 1.

Observe that if a matrix belongs to $\mathscr{M}'$, then its transpose not necessarily is in this class. For example, it is easy to check that the matrix

$$M = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

is an element of $\mathscr{M}'$, whereas M^T is not, since it does not satisfy Property 2. This observation suggests introducing two more classes of matrices that are more "symmetric." Denote by $\mathscr{M}_s'$ and $\mathscr{M}_s$ respectively the classes of matrices M for which both M and M^T belong to $\mathscr{M}'$ and $\mathscr{M}$. Obviously

$\mathcal{M}_s' \subset \mathcal{M}'$ and $\mathcal{M}_s \subset \mathcal{M}$; in both cases—in view of the above example—strict inclusion holds.

THEOREM 1. $\mathcal{M}' \subseteq \mathcal{M}$.

Proof. Properties 1 and 2 guarantee that if the actual tableau is not terminal, then the criss-cross method defines a pivot (does not **EXIT**). Since the number of the complementary bases is finite, one only has to show that cycling is not possible. If we assume to the contrary that cycling occurs, then we have a set J^* of indices of variables that entered or left the basis during cycling. Without loss of generality we can restrict our considerations to this index set J^*, and we may also assume that $n = \max\{i : i \in J^*\}$. Considering the cases when w_n enters and leaves the basis, we have one of the exclusive cases of Property 3 (see also Figure 1). Hence the finiteness of the criss-cross method is guaranteed by the orthogonality property and by Property 3. (Similar proofs—with more details—are presented also in [9, 12, 20, 21].)

As a consequence of this theorem, we have the inclusions $\mathcal{M}_s' \subset \mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{M}_s' \subseteq \mathcal{M}_s \subset \mathcal{M}$.

5. THE CRISS-CROSS METHOD AND SUFFICIENT LCP'S

Let us consider the class of sufficient matrices, denoted by $\mathcal{S}_u$. Note that $\mathcal{S}_u$ is a closed complete class.

THEOREM 2. $\mathcal{S}_u = \mathcal{M}_s'$.

Proof. We first prove that $\mathcal{S}_u \subset \mathcal{M}_s'$. A matrix is sufficient if and only if its transpose is sufficient, so it is enough to prove that the three properties hold for sufficient matrices.

Every principal transform (see Proposition 6) and any principal submatrix (see Proposition 3) of a sufficient matrix is sufficient, so if the required properties hold, then they hold for principal submatrices and principal pivotal transforms as well.

Property 1 follows from Proposition 4. Property 2 follows from Proposition 5. To prove Property 3 we only have to prove that cases **A** and **B** are exclusive. The others follow immediately from the orthogonality property.

Now let us assume to the contrary that for a sufficient LCP both cases **A** and **B** occur. Let the actual complementary solutions be denoted by (z, w) and (z', w') respectively. Without loss of generality we may assume that $z_n < 0$, $w_n = 0$, $z_n' = 0$, $w_n' < 0$, and all the other coordinates are nonnegative. Then using (1) and the sign and complementarity properties of vectors

(z, w) and (z', w'), we have

$$(Z - Z')M(z - z') = (Z - Z')(w - w') \leqslant 0.$$

For the nth coordinate we have $(z_n - z'_n)(w_n - w'_n) = -z_n w'_n < 0$, which contradicts the (column) sufficiency of matrix M.

On the other hand, if $M \in \mathscr{M}'_s$, then Properties 1 and 2 holds for M and M^T, and hence Propositions 7 and 8 imply that matrix M is sufficient. So the equivalence of $\mathscr{M}'_s$ and the class of sufficient matrices is proved. ■

REMARK. Note that as a consequence of Propositions 7 and 8 we have that if both M and M^T satisfy Properties 1 and 2, then M (and M^T) is sufficient. This proves that Property 3 is redundant for the definition of $\mathscr{M}'_s$.

As a consequence of Theorem 1 and Theorem 2 we have:

COROLLARY 1. *Let (q, M) be a given* LCP, *where q is an arbitrary vector and M is a sufficient matrix. Then the criss-cross method will process (q, M) in a finite number of steps.*

Now we are ready to formulate our main result.

THEOREM 3. $\mathscr{S}_u = M'_s = \mathscr{M}_s$.

Proof. Theorems 1 and 2 state that $\mathscr{S}_u = \mathscr{M}'_s \subset \mathscr{M}_s$. So one only has to prove that if a matrix M is not sufficient, then it does not belong to $\mathscr{M}_s$. If M is not sufficient, then Propositions 7 and 8 imply that either Property 1 or Property 2 does not hold, and so with a properly chosen vector q the criss-cross method cannot process the problem. ■

Summarizing our results, we have

$$\mathscr{S}_u = \mathscr{M}'_s \subseteq \mathscr{M}' \subset \mathscr{M},$$

$$\mathscr{S}_u = \mathscr{M}'_s = \mathscr{M}_s \subseteq \mathscr{M}.$$

It remains as an open question whether $\mathscr{M}'$ is equal to $\mathscr{M}$, or $\mathscr{M}'$ is a proper subset of $\mathscr{M}$.

Note added in proof: Recently our attention was called to the paper of R. W. Cottle and Y.-Y. Chang: *Least-Index resolution of Degeneracy in Linear Complementarity Problems with Sufficient Matrices*, SIAM Journal on Matrix Analysis and Applications 13(4):1131–1141, where ideas similar to ours are used in the context of the principal pivoting method.

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